# NON-MEAGER $P$-FILTERS ARE COUNTABLE DENSE HOMOGENEOUS

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Abstract. Let $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter that extends the Fréchet filter and identify $\mathcal{P}(\omega)$ with the Cantor set in the natural way. In this paper we prove that if $\mathcal{F}$ is a non-meager $P$-filter, then both $\mathcal{F}$ and ${}^{\omega}\mathcal{F}$ are countable dense homogeneous.

## 1. Introduction

A separable space $X$ is *countable dense homogeneous* (*CDH*) if every time $D$ and $E$ are countable dense subsets of $X$ there exists a homeomorphism $h : X \to X$ such that $h[D] = E$. Using the now well-known back-and-forth argument, Cantor gave the first example of a CDH space: the real line. Other examples of famous spaces that are in fact CDH are the Euclidean spaces, the Hilbert cube and the Cantor set. CDH spaces have motivated research resulting in a number of papers. See [4] for a small summary of past research and bibliography about CDH spaces.

In their Open Problems in Topology paper ([3]), Fitzpatrick and Zhou posed the following problems.

**1.1. Question** (Problem 6) Does there exist a CDH metric space that is not completely metrizable?

**1.2. Question** (Part 2 of Problem 4) For which 0-dimensional subsets $X$ of $\mathbb{R}$ is ${}^{\omega}X$ CDH?

Let us therefore restrict to separable metrizable spaces from this point on. Concerning these two problems, the following results have been obtained.

**1.3. Theorem** [4] Let $X$ be a separable metrizable space.

- If $X$ is CDH and Borel, then $X$ is completely metrizable.
- If ${}^{\omega}X$ is CDH, then $X$ is a Baire space.

**1.4. Theorem** [2] There is a CDH set of reals $X$ of size $\omega_1$ that is a $\lambda$-set and thus, not completely metrizable.

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Notice that the CDH space from Theorem 1.4 is not a Baire space so Theorem 1.3 indicates we cannot use this same technique to answer Question 1.2.

Another recent paper related to Questions 1.3 and 1.4 is [6], where the authors study ultrafilters as subspaces of the Cantor set. Recall that there is a natural bijection between the Cantor set and $\mathcal{P}(\omega)$ via characteristic functions. In this way we may identify $\mathcal{P}(\omega)$ with the Cantor set. Thus, any subset of $\mathcal{P}(\omega)$ can be thought of as a separable metrizable space. In this way, Medini and Milovich obtained the following result. For undefined terms about filters and ideals, see [1].

**1.5. Theorem** [6, Theorems 15, 21, 24, 41, 43 and 44] Assume $MA(countable)$. Then there exists a non-principal ultrafilter $\mathcal{U} \subset \mathcal{P}(\omega)$ with any of the following properties: $(a)$ $\mathcal{U}$ is CDH and a $P$-point, $(b)$ $\mathcal{U}$ is CDH and not a $P$-point, $(c)$ $\mathcal{U}$ is not CDH and not a $P$-point, and $(d)$ ${}^{\omega}\mathcal{U}$ is CDH.

Since ultrafilters do not even have the Baire property ([1, 4.1.1]), Theorem 1.5 gives us a consistent answer to Question 1.1 and consistent examples concerning Question 1.2.

The purpose of this paper is to extend these results on ultrafilters to a wider class of filters in the Cantor set. We have obtained the following result. Notice that it answers Questions 4, 5 and 10 of [6].

**1.6. Theorem** Let $\mathcal{F}$ be a non-meager $P$-filter on $\mathcal{P}(\omega)$ extending the Fréchet filter. Then both $\mathcal{F}$ and ${}^{\omega}\mathcal{F}$ are CDH.

It is also true that non-meager filters do not have the Baire property ([1, 4.1.1]). However, the existence of non-meager $P$-filters is still an open question (in ZFC). Nevertheless, it is known that if all $P$-filters are meager then there is an inner model with a large cardinal. See [1, 4.4.C] for a detailed description of this problem.

We also show that every CDH filter has to be non-definable in the following sense.

**1.7. Proposition** Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ extending the Fréchet filter. If one of $\mathcal{F}$ or ${}^{\omega}\mathcal{F}$ is CDH, then $\mathcal{F}$ is non-meager.

By Theorem 1.5 it is consistent that not all CDH filters are $P$-filters and that there exist non-CDH filters. An ideal situation would be to solve the following problems.

**1.8. Question** Give a nice combinatorial characterization of CDH filters.

**1.9. Question** Is there a CDH filter (ultrafilter) in ZFC? Is there a non-CDH and non-meager filter (ultrafilter) in ZFC?

## 2. Proofs of our results

For any set $X$, let $[X]^{<\omega}$ and $[X]^{\omega}$ denote the set of its finite and countable infinite subsets, respectively. Also ${}^{<\omega}X = \bigcup\{{}^{n}X : n < \omega\}$. The symbol $A - B$ will denote the set theoretic difference of $A$ minus $B$. Also, $A \triangle B$ denotes the symmetric difference of $A$ and $B$. Notice that $(\mathcal{P}(\omega), \triangle)$ is a topological group, it corresponds to addition modulo 2 in $({}^{\omega}2, +)$.

Given a filter $\mathcal{F} \subset \mathcal{P}(\omega)$, the dual ideal $\mathcal{F}^* = \{A \subset \omega : \omega - A \in \mathcal{F}\}$ is homeomorphic to $\mathcal{F}$ by means of the map that takes each subset of $\omega$ to its complement. So we may alternatively speak about a filter or its dual ideal. In particular, the following result is better expressed in the language of ideals. Its proof follows from [6, Lemma 20].

**2.1. Lemma** Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal, $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ a homeomorphism and $D$ a countable dense subset of $\mathcal{I}$. If there exists $x \in \mathcal{I}$ such that $\{d \triangle h(d) : d \in D\} \subset \mathcal{P}(x)$ , then $h[\mathcal{I}] = \mathcal{I}$.

Let $\mathcal{X} \subset [\omega]^\omega$. A tree $T \subset {}^{<\omega}([\omega]^{<\omega})$ is called a *$\mathcal{X}$-tree of finite subsets* if for each $s \in T$ there is $X_s \in \mathcal{X}$ such that for every $a \in [X_s]^{<\omega}$ we have $s^\frown a \in T$. It turns out that non-meager $P$-filters have a very useful combinatorial characterization as follows.

**2.2. Lemma** [5, Lemma 1.3] Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ that extends the Fréchet filter. Then $\mathcal{F}$ is a non-meager $P$-filter if and only if every $\mathcal{F}$-tree of finite subsets has a branch whose union is in $\mathcal{F}$.

Now we prove a combinatorial property that will allow us to construct autohomeomorphisms of the Cantor set that restrict to ideals. For $x \in \mathcal{P}(\omega)$, let $\chi(x) \in {}^\omega 2$ be its characteristic function.

**2.3. Lemma** Let $\mathcal{I}$ be a non-meager $P$-ideal that contains all finite subsets of $\mathcal{P}(\omega)$ and $D_0$, $D_1$ be two countable dense subsets of $I$. Then there exists $x \in \mathcal{I}$ such that

(i) for each $d \in D_0 \cup D_1$, $d \subset^* x$ and
(ii) for each $i \in 2$, $d \in D_i$, $n < \omega$ and $t \in {}^{n\cap x}2$, there exists $e \in D_{1-i}$ such that $d - x = e - x$ and $\chi(e)\restriction_{n\cap x} = t$.

*Proof.* Let $\mathcal{F}$ be the dual filter of $\mathcal{I}$ (so that $\mathcal{I} = \mathcal{F}^*$). We will construct an $\mathcal{F}$-tree of finite subsets $T$ and use Lemma 2.2 to find $x \in \mathcal{I}$ with the properties listed. Let us give an enumeration $(D_0 \cup D_1) \times {}^{<\omega}2 = \{(d_n, t_n) : n < \omega\}$ such that $\{d_n : n \equiv i \ (mod\ 2)\} = D_i$ for $i \in 2$.

The definition of $T$ will be by induction. For each $s \in T$ we also define $n(s) < \omega$, $F_s \in \mathcal{F}$ and $\phi_s : dom(s) \to D_0 \cup D_1$ so that the following properties hold.

(1) $\forall s, t \in T\ (s \subsetneq t \Rightarrow n(s) < n(t))$,
(2) $\forall s \in T\ \forall k < dom(s)\ (s(k) \subset n(s\restriction_{k+1}) - n(s\restriction_k))$,
(3) $\forall s, t \in T\ (s \subset t \Rightarrow F_t \subset F_s)$,
(4) $\forall s \in T\ (F_s \subset \omega - n(s))$,
(5) $\forall s, t \in T\ (s \subset t \Rightarrow \phi_s \subset \phi_t)$,
(6) $\forall s \in T,$ if $k = dom(s)$ $((d_{k-1} \cup \phi_s(k-1)) - n(s) \subset \omega - F_s)$.

Since $\emptyset \in T$, let $n(\emptyset) = 0$ and $F_\emptyset = \omega$. Assume we have $s \in T$ and $a \in F_s$, we have to define everything for $s^\frown a$. Let $k = dom(s)$. We start by defining $n(s^\frown a) = \max\{k, \max(a)\} + 1$. Next we define $\phi_{s^\frown a}$, we only have to do it at $k$ because of (5). We have two cases.

<u>Case 1</u>. There exists $m < dom(t_k)$ with $t_k(m) = 1$ and $m \in s(0) \cup \ldots \cup s(k-1)$. We simply declare $\phi_{s^\frown a}(k) = d_k$.

Case 2. Not Case 1. We define $r_{s^\frown a} \in {}^{n(s^\frown a)}2$ in the following way.

$$r_{s^\frown a}(m) = \begin{cases} d_k(m), & \text{if } m \in s(0) \cup \ldots \cup s(k-1) \cup a, \\ t_k(m), & \text{if } m \in dom(t_k) - (s(0) \cup \ldots \cup s(k-1) \cup a), \\ 1, & \text{in any other case.} \end{cases}$$

Let $i \in 2$ be such that $i \equiv k \ (mod\ 2)$. So $d_k \in D_i$, let $\phi_{s^\frown a}(k) \in D_{1-i}$ be such that $\phi_{s^\frown a}(k) \cap n(s^\frown a) = (r_{s^\frown a})^{\leftarrow}(1)$, this is possible because $D_{1-i}$ is dense in $\mathcal{P}(\omega)$. Finaly, define

$$F_{s^\frown a} = (F_s \cap (\omega - d_{k-1}) \cap (\omega - \phi_{s^\frown a}(k-1))) - n(s^\frown a).$$

Clearly, $F_{s^\frown a} \in \mathcal{F}$ and it is easy to see that conditions (1) - (6) hold.

By Lemma 2.2, there exists a branch $\{(y_0, \ldots, y_n) : n < \omega\}$ of $T$ whose union $y = \bigcup\{y_n : n < \omega\}$ is in $\mathcal{F}$. Let $x = \omega - y \in \mathcal{I}$. We prove that $x$ is the element we were looking for. It is easy to prove that (7) implies (i).

We next prove that (ii) holds. Let $i \in 2$, $n < \omega$, $t \in {}^{n\cap x}2$ and $d \in D_i$. Let $k < \omega$ be such that $(d_k, t_k) = (d, t')$, where $t' \in {}^n2$ is such that $t' \restriction_{n\cap x} = t$ and $t' \restriction_{n-x} = \underline{0}$. Consider step $k+1$ in the construction. Notice that we are in Case 2 of the construction and $r_{y\restriction_{k+1}}$ is defined. Then $\phi_{y\restriction_{k+1}}(k) = e$ is an element of $D_{1-i}$. It is not very hard to see that $d - x = e - x$ and $\chi(e) \restriction_{n\cap x} = t$. This completes the proof of the Lemma. □

**2.4. Lemma** If $\mathcal{F}$ is a non-meager $P$-filter, then ${}^\omega\mathcal{F}$ is homeomorphic to a non-meager $P$-filter.

*Proof.* Let

$$\mathcal{G} = \{A \subset \omega \times \omega : \forall n < \omega (A \cap (\omega \times \{n\}) \in \mathcal{F})\}.$$

It is easy to see that $\mathcal{G}$ is a filter on $\omega \times \omega$ and that it is a $P$-filter if $\mathcal{F}$ is. To see that $\mathcal{G}$ is non-meager if $\mathcal{F}$ is, use the characterization in [1, Theorem 4.1.2]. □

We now have everything cooked up to prove our results.

**Proof of Theorem 1.6.** By Lemma 2.4, it is enough to prove that $\mathcal{F}$ is CDH. Let $\mathcal{I} = \mathcal{F}^*$, it is enough to prove that $\mathcal{I}$ is CDH. Let $D_0$ and $D_1$ be two countable dense subsets of $\mathcal{I}$. Let $x \in \mathcal{I}$ be given by Lemma 2.3.

We will construct a homeomorphism $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $h[D_0] = D_1$ and

$$(\star) \qquad \forall d \in D \ (d \bigtriangleup h(d) \subset x).$$

By Lemma 2.1, $h[\mathcal{I}] = \mathcal{I}$ and we will have finished.

We shall define $h$ by approximations. By this we mean the following. We will give a strictly increasing sequence $\{n(k) : k < \omega\} \subset \omega$ and in step $k < \omega$ a homeomorphism (permutation) $h_k : \mathcal{P}(n(k)) \to \mathcal{P}(n(k))$ such that

$$(*) \qquad \forall r < s < \omega \ \forall a \in \mathcal{P}(n(k)) \ (h_s(a) \cap n(r) = h_r(a \cap n(r))).$$

By $(*)$, we can define $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ to be the inverse limit of the $h_k$, which is a homeomorphism.

Let $D_0 \cup D_1 = \{d_n : n < \omega\}$ in such a way that $\{d_n : n \equiv i \ (mod\ 2)\} = D_i$ for $i \in 2$. To make sure that $h[D_0] = D_1$, in step $k$ we have to decide the value of $h(d_k)$ when $k$ is even and the value of $h^{-1}(e_k)$ when $k$ is odd. We do this by approximating a bijection $\pi : D_0 \to D_1$ in $\omega$ steps. In step $s < \omega$, we would like to have $\pi$ defined in some finite set so that we have the following conditions:

$(a)_s$ if $r < s$ is even, then $h_s(d_r \cap n(s)) = \pi(d_r) \cap n(s)$, and
$(b)_s$ if $r < s$ is odd, then $h_s(d_r \cap n(s)) = \pi^{-1}(d_r) \cap n(s)$.

Clearly $\forall s < \omega((a)_s \wedge (b)_s) \Leftrightarrow h[D] = E$. As we do the construction, we need to take care that the following technical condition holds.

$$(c)_s \quad \forall k \in n(s) - x\ \forall a \in \mathcal{P}(n(s))\ (k \in a \Leftrightarrow k \in h_s(a))$$

We also require that $\pi$ satisfy the following condition that implies $(\star)$.

$$(\star\star) \qquad \forall d \in D_0\ (d - x = \pi(d) - x).$$

Assume that we have defined $n(0) < \ldots < n(s-1)$, $\pi \cup \pi^{-1}$ in $\{d_k : k < s\}$ and $h_0, \ldots, h_{s-1}$ in such a way that $(a)_{s-1}$, $(b)_{s-1}$ and $(c)_{s-1}$ hold. Let us consider the case when $s$ is even, the other case is treated in a similar fashion.

If $d_s = \pi^{-1}(d_k)$ for some odd $k < s$, let $n(s) = n(s-1)+1$. It is easy to define $h_s$ so that it is compatible with $h_{s-1}$ in the sense of $(*)$ and in such a way that $(a)_s$, $(b)_s$ and $(c)_s$ hold. So we may assume this is not the case.

Notice that the set $S = \{d_k : k < s+1\} \cup \{\pi(d_k) : k < s, k \equiv 0\ (mod\ 2)\} \cup \{\pi^{-1}(d_k) : k < s, k \equiv 1\ (mod\ 2)\}$ is finite. Choose $p < \omega$ so that $d_s - x \subset p$. Let $r_0 = h_{s-1}(d_s \cap n(s-1)) \in \mathcal{P}(n(s-1))$. Choose $n(s-1) < m < \omega$ and $t \in {}^{m\cap x}2$ in such a way that $t^{\leftarrow}(1) \cap n(s-1) = r_0 \cap n(s-1) \cap x$ and $t$ is not extended by any element of $\{\chi(a) : a \in S\}$. By Lemma 2.3, there exists $e \in E$ such that $d_s - x = e - x$ and $\chi(e) \restriction_{m\cap x} = t$. Notice that $e \notin S$ and $\chi(e) \restriction_{n(s-1)} = r_0$. We define $\pi(d_s) = e$. Notice that $(\star\star)$ holds for $d_s$.

Now that we have chosen $\pi(d_s)$, let $n(s) > \max(p, m)$ be such that there are no two distinct $a, b \in S \cup \{\pi(d_s)\}$ with $a \cap n(s) = b \cap n(s)$. Topologically, all elements of $S \cup \{\pi(d_s)\}$ are contained in distinct basic open sets of size $1/n(s)$.

We finally define the bijection $h_s : \mathcal{P}(n(s)) \to \mathcal{P}(n(s))$. For this part of the proof we will use characteristic functions instead of subsets of $\omega$ (otherwise the notation would become cumbersome). Therefore, we may say $h_k : {}^{n(k)}2 \to {}^{n(k)}2$ is a homeomorphism for $k < s$. Recall that we want to preserve propertes $(*)$ and $(c)_s$. Notice that for each pair $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)-x}2$ of compatible functions we have that $(h_{s-1}(q), q')$ are compatible by $(c)_{s-1}$. Thus, in the construction we may ask that

$$(\triangledown) \qquad \forall a \in {}^{n(s)}2\ (q \cup q' \subset a \Leftrightarrow h_{s-1}(q) \cup q' \subset h_s(a)).$$

Notice that $\triangledown$ implies $(*)$ and $(c)_s$. So for each pair $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)-x}2$ of compatible functions we only have to find a bijection $g : {}^{T}2 \to {}^{T}2$, where $T = (n(s) \cap x) - n(s-1)$ (that may depend on such pair) and define $h_s : {}^{n(s)}2 \to {}^{n(s)}2$ as

$$h_s(a) = h_{s-1}(q) \cup q' \cup g(f \restriction_T),$$

whenever $a \in {}^{n(s)}2$ and $q \cup q' \subset a$. There is only one restriction in the definition of $g$ and it is imposed by conditions $(a)_s$ and $(b)_s$; namely that $g$ is compatible with the bijection $\pi \restriction_{S\cap D_0}$. But by the choice of $n(s)$ this is easy to do. Thus, $(a)_s$ and $(b)_s$ hold. This finishes the inductive step and the proof.

□

**Proof of Proposition 1.7.** Let $X \in \{\mathcal{F}, {}^{\omega}\mathcal{F}\}$ and assume $X$ is CDH. If $\mathcal{F}$ is the Fréchet filter, then $\mathcal{F}$ is countable and cannot be CDH. If $\mathcal{F}$ is not the Fréchet filter, there exists $x \in \mathcal{F}$ such that $\omega - x$ is infinite. Thus, $\{y : x \subset y \subset \omega\}$ is a copy of the Cantor set contained in $\mathcal{F}$. Further, ${}^{\omega}\mathcal{F}$ always contains a copy of the Cantor set. So it is always true that $X$ contains a copy $C$ of the Cantor set.

Assume that $\mathcal{F}$ is meager, let us arrive to a contradiction. First, let us prove that ${}^{\omega}\mathcal{F}$ is also meager. Notice that $\mathcal{F}^*$ is a topological subgroup of $(\mathcal{P}(\omega), \triangle)$ so ${}^{\omega}\mathcal{F}$ is homogeneous. So, assuming ${}^{\omega}\mathcal{F}$ is not meager, then it is a Baire space by [6, Proposition 3]. Let $\pi : {}^{\omega}\mathcal{F} \to \mathcal{F}$ be the projection to the first coordinate, then $\pi$ is an open map. It is easy to see that this implies that $\mathcal{F}$ is a Baire space. This contradiction implies that ${}^{\omega}\mathcal{F}$ is meager.

So we have that $X$ is a meager-in-itself space that contains a Cantor set $C$. Let $D \subset X$ be a countable dense subset of $X$ such that $D \cap C$ is dense in $C$. By [4, Lemma 2.1], it is possible to find a countable dense subset $E$ of $X$ that is a $G_\delta$ set relative to $X$. Let $h : X \to X$ be a homeomorphism such that $h[D] = E$. Then $h[D \cap C]$ is a countable dense subset of the Cantor set $h[C]$ that is a relative $G_\delta$ subset of $h[C]$, this is impossible. This contradiction shows that $\mathcal{F}$ is non-meager and completes the proof. $\square$

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